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THE UNSTEADY MICROROTATIONAL REGIME†

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Using the Cossera model, in which the internal energy depends linearly on the microrotation gradients and non-linearly on the microrotations themselves, the propagation of small isothermal perturbations is investigated. It is shown that for a steady flow of the medium, an unsteady microrotational regime is possible. This effect is interpreted as acoustic emission by the medium.

1. LET t BE the time and x_i Cartesian coordinates in some frame of reference. Latin indices correspond to coordinates and take values from 1 to 3.

The state of a viscous medium with a microstructure is described by the density field $\rho=\rho(t, x_i)$, velocity field $\upsilon_j=\upsilon_j(t, x_i)$, angles of microrotations about the coordinate axes $\varphi_j=\varphi_j(t, x_i)$, and temperature $T=T(t, x_i)$. These fields satisfy the dynamical equations [1] of continuity, momentum, angular momentum, and energy

$$\frac{d\rho}{dt}+\rho\upsilon_{i,i}=0 \tag{1.1}$$

$$\rho\frac{d\upsilon_i}{dt}=\sigma_{ij,j}+f_i \tag{1.2}$$

$$\frac{d}{dt}\left[\varepsilon_{ijk}x_j\rho\upsilon_k+J\frac{d\varphi_i}{dt}\right]+\left[\varepsilon_{ijk}x_j\rho\upsilon_k+J\frac{d\varphi_i}{dt}\right]\upsilon_{l,l}= \tag{1.3}$$

$$=\left[\varepsilon_{ikl}x_k\sigma_{lj}+\pi_{ij}\right]_{,j}+\varepsilon_{ijk}x_jf_k+m_i+M_i$$

$$\frac{d}{dt}[K+U]+[K+U]\upsilon_{i,i}= \tag{1.4}$$

$$=[\upsilon_i\sigma_{ij}]_{,j}+\left[\frac{d\varphi_i}{dt}\pi_{ij}\right]_{,j}+f_i\upsilon_i+m_i\frac{d\varphi_i}{dt}-q_{i,i}+\varepsilon$$

$$K=\frac{1}{2}\rho\upsilon_i\upsilon_i+\frac{1}{2}J\frac{d\varphi_i}{dt}\frac{d\varphi_i}{dt},\quad \frac{d}{dt}=\frac{\partial}{\partial t}+\upsilon_i\frac{\partial}{\partial x_i}$$

Here σ_{ij} is the stress tensor, π_{ij} is the angular momentum stress tensor, f_i are the external forces, m_i is the moment of external forces, M_i is the moment of internal forces, J is the moment of inertia density, which we shall take to be constant, K is the kinetic energy density, U

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is the internal energy density, q_i is the thermal flux, and ε is the rate of heat production per unit volume. We shall also use the Clausius–Duhem inequality

$$\rho\frac{ds}{dt}-T^{-1}\varepsilon+\left(q_iT^{-1}\right)_{,i}\geqslant 0 \tag{1.5}$$

We shall assume that the internal energy U depends on the parameters ρ, T, φ_i^0, $\psi_{ij}=\varphi_{i,j}$ where

$$\varphi_i^0=\varphi_i-\frac{1}{2}\varepsilon_{ijk}u_{k,j},\quad \frac{d}{dt}u_i=\upsilon_i$$

Then the second law of thermodynamics for particles of the medium has the form

$$Tds=d(\rho^{-1}U)+pd\rho^{-1}+\Phi\rho^{-1}d\varphi_i^0+\Phi_{ij}\rho^{-1}d\psi_{ij} \tag{1.6}$$

where p is the pressure, and Φ_i and Φ_{ij} are thermodynamic forces associated with the microstructure of the porous medium.

We will define the elastic potential as follows:

$$W=\rho^{-1}\left(U-U\Big|_{\varphi_i^0=0,\psi_{ij}=0}\right)$$

Then according to (1.6) we have

$$\Phi_i=-\left(\frac{\partial W}{\partial\varphi_i^0}\right)_{\rho,s},\quad \Phi_{ij}=-\left(\frac{\partial W}{\partial\psi_{ij}}\right)_{\rho,s} \tag{1.7}$$

The state of the medium at $\varphi_i^0=0$ will be energetically stable if the potential W has a local minimum at $\varphi_{ij}=0$, $\psi_{ij}=0$ If we confine ourselves to a quadratic approximation, then

$$W=\frac{1}{2}\varsigma\varphi_i^0\varphi_i^0+\frac{1}{2}\lambda_1(\psi_{ii})^2+\frac{1}{2}\lambda_2\psi_{ij}^s\psi_{ij}^s+\frac{1}{2}\lambda_s\psi_{ij}^a\psi_{ij}^a \tag{1.8}$$

where

$$\varsigma>0,\ \lambda_\alpha>0,\ \psi_{ij}^s=\psi_{(ij)}-\frac{1}{3}\delta_{ij}\psi_{kk},\ \psi_{ij}^a=\psi_{[ij]}$$

From (1.1)–(1.4) and (1.6) it is easy to compute the expression specifying the entropy change

$$\rho T\frac{ds}{dt}=[\sigma_{ij}+p\delta_{ij}]\upsilon_{i,j}+\left[\pi_{ijh}\left(\frac{d\varphi_i}{dt}\right)_{,j}+\Phi_{ij}\frac{d\psi_{ij}}{dt}\right]+ \tag{1.9}$$

$$+[\Phi_i-M_i]\frac{d\varphi_i}{dt}+\varepsilon_{ijk}\sigma_{jk}\frac{d\varphi_i}{dt}+\varepsilon-q_{i,i}$$

The viscous stress tensor $\tau_{ij}=\sigma_{ij}+p\delta_{ij}$ consists of symmetric $\tau_{ij}^s=\tau_{(ij)}$ and antisymmetric $\tau_{ij}^a=\tau_{[ij]}$ parts.

We shall consider processes for which $T=\text{const}$ and $\varepsilon=0$.

From (1.5) and (1.9) we have the inequality

$$\Sigma dt \geqslant 0 \tag{1.10}$$

$$\Sigma = \tau_{ij}^s \upsilon_{(i,j)} + \left[\pi_{ij}\left[\frac{d\varphi_i}{dt}\right]_{i,j} + \Phi_{ij}\frac{d\psi_{ij}}{dt}\right] +$$
$$+[\Phi_i - M_i]\frac{d\varphi_i}{dt} + \left[\upsilon_{[j,k]} + \varepsilon_{ijk}\frac{d\varphi_i}{dt}\right]\tau_{jk}^a$$

Inequality (1.10) is satisfied in the quadratic approximation if the following constitutive relations hold

$$\tau_{ij}^s = \eta\delta_{ij}\upsilon_{k,k} + 2\mu\left[\upsilon_{(i,j)} - \frac{1}{3}\delta_{ij}\upsilon_{k,k}\right] \tag{1.11}$$

$$\pi_{ij} = -\Phi_{ij} + \Lambda_1\delta_{ij}\frac{\partial\psi_{kk}}{\partial t} + 2\Lambda_2\frac{\partial}{\partial t}\left[\psi_{(ij)} - \frac{1}{3}\delta_{ij}\psi_{kk}\right] + 2\Lambda_3\frac{\partial}{\partial t}\psi_{[i,j]} \tag{1.12}$$

$$M_i = \Phi_i - \alpha\frac{\partial\varphi_i}{\partial t} \tag{1.13}$$

$$\tau_{ij}^a = 2\beta\left[\upsilon_{[i,j]} + \varepsilon_{ijk}\frac{\partial\varphi_k}{\partial t}\right] \tag{1.14}$$

Here η is the bulk viscosity, μ is the shear viscosity, and Λ_α, and α and β are non-negative dissipative coefficients. We will further restrict ourselves to the case when η, μ, $\beta > 0$, $\Lambda_\alpha = 0$, $\alpha = 0$.

2. We will investigate the propagation of small isothermal perturbations on an inhomogeneous background $\rho = \rho_0$, $\upsilon_i = 0$, $\varphi_i = 0$, putting $c^2 = (\partial p/\partial\rho)_t$, $r = \rho - \rho_0$. Then it follows from Eqs (1.1)–(1.3), (1.8), (1.9), (1.11)–(1.14) that

$$\frac{\partial r}{\partial t} + \rho_0\upsilon_{i,i} = 0$$

$$\rho_0\frac{\partial\upsilon_i}{\partial t} + c^2 r_{,i} - \left(\eta + \frac{1}{3}\mu - \beta\right)\upsilon_{k,ki} - (\mu + \beta)\upsilon_{i,kk} + 2\beta\varepsilon_{ijk}\frac{\partial}{\partial t}\varphi_{j,k} = 0 \tag{2.1}$$

$$J\frac{\partial^2\varphi_i}{\partial t^2} + 2\beta\left[\varepsilon_{ijk}\upsilon_{j,k} + 2\frac{\partial\varphi_i}{\partial t}\right] -$$
$$-\left[\lambda_1 + \frac{1}{6}\lambda_2 - \frac{1}{2}\lambda_3\right]\varphi_{k,ki} - \frac{1}{2}[\lambda_2 + \lambda_3]\varphi_{i,kk} + \varsigma\varphi_i^0 = 0$$

It is convenient to solve Eqs (2.1) by the Fourier transformation method, with each unknown function $f = f(t, x_i)$ replaced by its transform

$$f_F(\omega, n_j) = \int e^{-i(\omega t + n_j x_j)} f(t, x_j)\,dt\,dx_1 dx_2 dx_3$$

System (1.5) turns into a system of linear equations whose determinant is given by

$$\det A = P_1 P_2 P_3^2$$

$$P_1 = i\rho_0\omega^2 - i\rho_0 c^2 n_k n_k + \left(\eta + \frac{4}{3}\mu\right)\omega n_k n_k$$

$$P_2 = -J\omega^2 + 4\beta i\omega + \left(\lambda_1 + \frac{2}{3}\lambda_2\right)n_k n_k + \varsigma$$

$$P_3 = (i\rho_0\omega + (\mu+\beta)n_k n_k)\left(-J\omega^2 + 4\beta i\omega + \frac{1}{2}(\lambda_2+\lambda_3)n_k n_k + \varsigma\right) -$$

$$-4\beta(\beta + \varsigma(4i\omega)^{-1})i\omega n_k n_k$$

Thus the dispersion surface decomposes into three surfaces $P_\alpha = 0$ $(a = 1, 2, 3)$.

The dispersion relation $P_1 = 0$ corresponds to longitudinal motion of the medium, which decouples from rotational degrees of freedom.

The dispersion relation $P_2 = 0$ corresponds to longitudinal-rotational waves, which decouple from translational degrees of freedom. At high frequencies these waves have the asymptotic form

$$n = -\omega\upsilon^{-1} + \gamma i + O(\omega^{-1}) \tag{2.2}$$

$$\upsilon = \left(\lambda_1 + \frac{2}{3}\lambda_2\right)^{1/2} J^{-1/2}, \quad \gamma = 2\upsilon\left(\lambda_1 + \frac{2}{3}\lambda_2\right)^{-1}\beta$$

where n is the wave number, ν is the wave velocity, and γ is the decrement.

The dispersion relation $P_3 = 0$ corresponds to transverse coupled translational-rotational motions. For the wave number n there is a biquadratic equation, one of whose solutions also has the asymptotic form (2.2) at high frequencies, with

$$\upsilon = 2^{-1/2}(\lambda_2+\lambda_3)^{1/2} J^{-1/2}, \quad \gamma = 4\upsilon(\lambda_2+\lambda_3)^{-1}\beta\mu(\mu+\beta)^{-1}$$

3. Actual rocks generate acoustic signals over a broad frequency band when they are under stress. The original acoustic impulse, propagating from the generating region towards the receiver, evolves and acquires a different amplitude-frequency behaviour. It has traditionally been assumed that the mechanism by which the original acoustic impulse is generated is associated either with defects in microcrystals constituting the material or with dislocations, twinning, etc., or with crack formation [2]. However, within the framework of the non-linear theory of elasticity for media with microstructures [3] a slow translational motion can also lead to ultrasound generation as a result of resonance between the low- and high-frequency branches of the dispersion surface. In this section it is shown that the original acoustic impulse can also appear during the creep of materials with microstructure.

We shall interpret the presence of an unsteady (high-frequency) microrotational regime during slow translational motion of the medium as the generation of the original acoustic impulse. In accordance with the ideas of [3], in order to describe such an effect it is natural to consider the non-linear theory of a medium with microstructure. We will take account of the non-linearity by specifying a more-complex expression for W than (1.8), replacing $\zeta\varphi_i^0\varphi_i^0/2$ with a smooth function w depending on the microrotations φ_i^0.

We will investigate a class of transverse translational-rotational motions. We put

$$\upsilon_i = \delta_{12}\upsilon, \quad \varphi_i = \delta_{13}\varphi, \quad \upsilon = \upsilon(t,x), \quad \varphi = \varphi(t,x), \quad \upsilon = \frac{\partial}{\partial t}u, \quad x = x_1.$$

Then from Eqs (1.2), (1.3), (1.8), (1.13)–(1.16) and the new expression for W we obtain the equation

$$J\varphi_{tt} + 4\beta\varphi_t - \Lambda\varphi_{xx} + \frac{\partial w}{\partial\varphi} = F \tag{3.1}$$

$$\Lambda = \frac{1}{2}(\lambda_1 + \lambda_2), \; F = 2\beta\upsilon_x + \frac{1}{2}\Lambda u_{xxx}$$

We shall treat the right-hand side of Eq. (3.1) as a source, or as an external field. Suppose this source is stationary, i.e. $F = F(x)$. Then Eq. (3.1) has a unique stationary solution $\varphi_0 = \varphi_0(x)$ satisfying zero boundary conditions as $x \to \pm\infty$. We shall consider the local stability of this solution.

Let $\Phi = \varphi - \varphi_0$ be a small perturbation of the stationary solution. Equation (3.1) reduces to a dynamic equation for Φ (where H is the Schrödinger operator)

$$J\Phi_{tt} + 4\beta\Phi_t + H\Phi = 0 \tag{3.2}$$

$$H = -\Lambda\partial^2/\partial k^2 + V, \quad V = V(x) = (\partial^2 w/\partial x^2)\big|_{\varphi=\varphi_0(x)}$$

Since any function $\Phi = \Phi(t, x)$ can be decomposed into eigenfunctions of the Schrödinger operator [4] with certain coefficients $C = C(t, \lambda)$ (where λ is the eigenvalue), it is sufficient to consider the equation

$$JC_{tt} + 4\beta C_t + \lambda C = 0$$

following from Eq. (3.2). This equation has the solution

$$C = C_1 e^{i\omega_1 t} + C_2 e^{i\omega_2 t}, \quad \omega_{1,2} = J^{-1}(2\beta i \pm (J\Lambda - 4\beta^2)^{1/2})$$

We note that for $\lambda < 0$ an exponentially growing mode exists, and the solution $\varphi_0(x)$ is unstable. If, however, the spectrum of the Schrödinger operator H is non-negative, the solution φ_0 is stable.

We consider in more detail the conditions under which the Schrödinger operator H can have negative spectral points. Since the operator $(-\Lambda\partial^2/\partial x^2)$ is positive-definite, negative spectral points are only possible when the potential V takes negative values.

Thus if the function w is convex, the stationary solution $\varphi_0(x)$ is stable. We will now show that if the potential w has non-convex parts, flows of the medium exist in which the stationary solution $\varphi_0(x)$ is unstable. In this case, for initial conditions of general form the microrotational regime will be unsteady.

Thus, suppose $V_* = (\partial^2 w/\partial\varphi^2)|_{\varphi=\varphi_*} < 0$, and suppose that L and ε are positive quantities with dimensions of length such that $\varepsilon/L \ll 1$. We define

$$\varphi_0(x) = 0 \quad \text{for} \quad x \in (-\infty, 0] \cup [L, +\infty)$$

$$\varphi_0(x) = \varphi_* \quad \text{for} \quad x \in [\varepsilon, L-\varepsilon]$$

and let φ_0 be a smooth monotonic function in the intervals $[0, \varepsilon]$ and $[L-\varepsilon, L]$.

We determine the flow of the medium from the equation

$$-\Lambda\partial^2\varphi_0/\partial x^2 + (\partial^2 w/\partial x^2)\big|_{\varphi=\varphi_0(x)} = 2\beta\upsilon_x$$

In this case, the solution of the spectral problem for the Schrödinger operator, apart from terms of order ε/L, is close to the solution of the standard quantum mechanical problem for a rectangular potential well [5]

$$V(x) = V_0 = (\partial^2 w/\partial x^2)\big|_{\varphi=0} \quad \text{for} \quad x \in (-\infty, 0] \cup [L, +\infty)$$

and $V(x) = V_*$ for $x \in [0, L]$. In the well problem the Schrödinger operator has a negative spectrum, the lowest point of which can be as close to V_* as desired for sufficiently large L.

Thus it has been shown that for a non-convex function w there are steady flows of the medium for which there is an unsteady process at the level of medium particle microrotations. This phenomenon has a natural interpretation as the generation of the initial acoustic impulse.

We note that a non-stationary solution $\varphi = \varphi(t, x)$ has a range of completely determined properties (such as the spectrum) which do not depend on the initial conditions, but on the parameters J, β and the form of the function w. This is due to the fact that for equations of the form (3.1) the existence of a compact attractor with finite Hausdorff dimension [6,7] has been proved. Hence the dynamics of microrotations will be completely defined by the structure of this attractor. Thus the acoustic emission has the form of deterministic noise.

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